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On a class of embeddings of massive Yang-Mills theory

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A power-counting renormalizable model into which massive Yang-Mills theory is embedded is analyzed. The model is invariant under a nilpotent BRST differential s. The physical observables of the embedding theory, defined by the cohomology classes of s in the Faddeev-Popov neutral sector, are given by local gauge-invariant quantities constructed only from the field strength and its covariant derivatives.

1 Introduction

In the search for a renormalizable model of massive Yang-Mills theory it has become apparent since a long time that at least one additional scalar field is required in order to fulfill physical unitarity already at tree level [1, 2, 3, 4, 5].

The spontaneous breaking of gauge symmetry provides a way to account for massive non-Abelian gauge bosons within the class of power-counting renormalizable models by means of the Higgs mechanism (see e.g. [14]). In its simplest version one additional physical scalar particle is introduced in the spectrum. The corresponding field acquires a non-vanishing vacuum expectation value (*v.e.v.*) due to the presence of a suitably designed quartic potential, which triggers the spontaneous gauge symmetry breaking.

On the other hand, in the BRST quantization of gauge theories [7, 8, 9] the requirement of physical unitarity is translated into the nilpotency of the relevant BRST differential, together with the condition of the absence of quantum anomalies for the corresponding Slavnov-Taylor (ST) identities [10, 11, 12]. Both requirements are fulfilled in the models based on the Higgs mechanism [13].

In this paper we would like to address an alternative method to generate a mass for non-Abelian gauge bosons, based on a fixing condition for an additional shift symmetry of a larger model, into which massive Yang-Mills theory can be embedded.

We wish to study whether it is possible to construct an embedding theory, with a suitably chosen field content, which is invariant under a nilpotent BRST differential s such that all the fields, besides the gauge fields A^a_μ and their gauge ghost fields ω^a, pair into BRST doublets with their shift ghost partners. As a consequence, these additional fields do not affect the cohomology classes of s, which reduce in the sector with zero Faddeev-Popov (FP) charge (physical observables) to gauge-invariant quantities constructed only from the field strength and its covariant derivatives.

A first possibility to explore in order to find the correct embedding model for massive Yang-Mills theory is to apply the embedding procedure to the field content of the ordinary Higgs model in the presence of a single Higgs multiplet. As we will show, the extension necessarily contains in the bosonic sector a scalar field ϕ and a vector field b_μ. ϕ is to be identified with the scalar field appearing in the usual Higgs mechanism. The component of ϕ which acquires a non-zero *v.e.v.* is in correspondence with the standard Higgs field.

The resulting model turns out to have good UV properties, but it shows a violation of the IR power-counting conditions which destroys renormalizability by power-counting. As a consequence, the embedding procedure in the presence of the Higgs mechanism fails to produce the sought-for renormalizable embedding of massive Yang-Mills theory. Nevertheless, from the analysis of the Higgs model embedding it can be readily seen, on the basis of cohomological arguments, that a different field content leads to a possible successful embedding of massive Yang-Mills theory. In this minimal embedding model the field ϕ is no longer needed and only the additional field b_μ appears in the bosonic sector together with the gauge fields A^a_μ.

The minimal embedding model fulfills both the UV and the IR power-counting conditions and is therefore power-counting renormalizable. It also shows a simple cohomological structure in the extended field sector.

The most important feature of the minimal embedding model is that spontaneous gauge symmetry breaking is not realized in it and still the physical spectrum contains massive gauge fields. The set of the physical observables of the theory is given by the local gauge-invariant quantities, constructed only from the field strength and its covariant derivatives. Hence the minimal embedding model could be considered as a candidate for a renormalizable theory of massive non-Abelian gauge bosons.

At the diagrammatic level it can be seen that the physical scalar required in the unitarity diagrams [6] is now provided by ∂b. It should be noted however that all the properties of the minimal embedding model are actually a direct and natural consequence of its BRST symmetry and can be better understood on purely cohomological grounds. One can show that the relevant ST identities are anomaly-free and therefore they can be restored order by order in perturbation theory by a suitable choice of finite action-like counterterms [16, 17, 18, 19]. This in turn guarantees the consistent quantization of the model to all orders in the loop expansion.

In this paper we focus on renormalizability and on the off-shell cohomological properties of the theory, with the aim of identifying the local physical observables. Moreover we also propose a method to characterize the relevant asymptotic states, by making use of techniques developed to study the on-shell cohomology of gauge theories [9, 11].

Many more things remain to be done. The observables of the model could be computed in perturbation theory and compared with the corresponding quantities in other models for massive gauge bosons. The construction pro-

posed in this paper might also suggest different alternatives in order to obtain a candidate for massive Yang-Mills theory. The question of the inclusion of massive fermions within the embedding formalism should also be analyzed, in view of possible phenomenological applications of this class of theories.

The paper is organized as follows. In Sect. 2 we analyze the embedding of the ordinary Higgs model with a single Higgs multiplet and discuss the associated embedding BRST differential. We study the UV and IR properties of the resulting theory and show that it fails to be renormalizable by power-counting, due to a bad behaviour in the IR regime.

In Sect. 3 we move to the analysis of the minimal embedding procedure and show that it leads to a well-defined power-counting renormalizable theory. For definiteness we work with the gauge group $SU(2)$, but the construction can be straightforwardly extended to other groups of interest, including those involving Abelian factors like $SU(2) \times U(1)$. The physical observables of this model, as selected by the cohomology of the relevant BRST differential s in the FP-neutral sector, are the gauge-invariant quantities constructed from the field strength and its covariant derivatives. Moreover, we analyze the structure of the asymptotic states and show that they contain three massive $SU(2)$ gauge bosons. Their physical polarizations are selected by the linearized BRST differential $\hat{s}$ to be the expected transverse ones. Finally conclusions are given in Sect. 4.

Appendices A and B contain the detailed construction of the Hilbert spaces needed in order to identify the physical asymptotic states of the minimal embedding model, while the UV and IR degrees of the fields are summarized in Appendix C.

2 The embedding procedure in the presence of the Higgs mechanism

As pointed out in the introduction, an extended set of fields which pair into BRST doublets is needed in order to construct an embedding model for massive Yang-Mills theory, while guaranteeing the nilpotency of the relevant BRST differential.

Since the new fields will form BRST doublets, they do not affect the cohomology of the model [15, 16, 20]. As a consequence, the physical observables of the embedding theory, defined by the cohomology classes of the

relevant BRST differential in the FP-neutral sector, coincide with the gauge-invariant quantities constructed only from the field strength and its covariant derivatives.

A first possibility to explore in order to find the correct embedding model for massive Yang-Mills theory is to apply this procedure to the field content of the ordinary Higgs model in the presence of a single Higgs multiplet. If power-counting renormalizability is dropped, the Stückelberg model [22] and its generalizations [23] could also be considered. However, in this paper we aim at finding a power-counting renormalizable embedding for massive Yang-Mills theory, and we will not discuss such alternatives.

As we will show, the extension necessarily contains in the bosonic sector a scalar field ϕ and a vector field b_μ. ϕ is to be identified with the scalar field appearing in the usual Higgs mechanism. The component of ϕ which acquires a non-zero *v.e.v.* is in correspondence with the standard Higgs field.

In the embedding of the Higgs model we require that the complex scalar ϕ forms a doublet pair together with its BRST partner h. We denote by T^a the generators of the Lie algebra $\mathfrak{g}$ acting on the representation to which ϕ, h belong. The commutation relations are

$$[T^a, T^b] = i f^{abc} T^c \,. \tag{1}$$

The normalization is chosen in such a way that

$$Tr(T^a T^b) = 2\delta^{ab} \,. \tag{2}$$

The covariant derivative on ϕ is

$$D_\mu \phi(x) = \partial_\mu \phi(x) - i A_{\mu a} T^a \phi(x) \tag{3}$$

and analogously for h. The BRST differential s acts on ϕ, h as follows [1]

$$\begin{aligned} & s\phi = i\omega_a T^a \phi + h \,, \qquad s\phi^\dagger = -i\omega_a \phi^\dagger T^a + h^\dagger \,, \\ & sh = i\omega_a T^a h \,, \qquad sh^\dagger = -i\omega_a h^\dagger T^a \,. \end{aligned} \tag{4}$$

ω^a are the ghost fields associated to the gauge transformation of the gauge group G whose Lie algebra is $\mathfrak{g}$. The BRST transformation of the gauge field A^a_μ is

$$s A^a_\mu = (D_\mu \omega)^a = \partial_\mu \omega^a + f^{abc} A^b_\mu \omega^c \,, \tag{5}$$

[1]Strictly speaking, in eq.(4) the BRST doublet is formed by $(\phi, i\omega_a T^a \phi + h)$. We will comment later on on the decomposition of the full BRST differential s into its gauge part and its pure shift part.

while the BRST transformation of the ghost fields is

$$s\omega^a = -\frac{1}{2} f^{abc} \omega^b \omega^c \,. \tag{6}$$

$\phi, \phi^\dagger$ are commuting, $h, h^\dagger$ and ω^a are anticommuting.

The Faddeev-Popov (FP) charge is assigned as follows:

$$\mathrm{FP}(\phi) = \mathrm{FP}(\phi^\dagger) = 0 \,, \qquad \mathrm{FP}(h) = \mathrm{FP}(h^\dagger) = 1 \,. \tag{7}$$

The ghost fields ω^a have FP charge +1.

Since ϕ is a component of a BRST doublet it follows from general cohomological arguments [15, 16, 20] that in any action invariant under the BRST differential s all terms depending on ϕ must be s-exact. Since we wish to include in the classical action of the embedding of the Higgs model the term $(D_\mu \phi)^\dagger D^\mu \phi$, we need in the sector with FP-charge -1 an anticommunting antighost field ψ^μ in the same representation as ϕ.

This also implies that a new ghost field ξ_μ, pairing with the antifield ψ_μ, will enter into the field content of the embedding model. ξ_μ is in the same representation as ϕ and has FP-charge +1. It forms a BRST doublet with a FP-neutral field b_μ, again in the same representation as ϕ.

We assign the following transformation rules for ψ^μ:[2]

$$\begin{aligned} s\psi^\mu &= +i\omega_a T^a \psi^\mu + D^\mu \phi + D_\rho b^{\rho\mu} \,, \\ s\psi^{\mu\dagger} &= -i\omega_a \psi^{\mu\dagger} T^a - (D^\mu \phi)^\dagger - (D_\rho b^{\rho\mu})^\dagger \,. \end{aligned} \tag{8}$$

In the above equation we have set

$$b_{\rho\mu} = D_{[\rho} b_{\mu]} \,, \tag{9}$$

where b_μ is a complex field whose BRST transformation is given by

$$sb_\mu = +i\omega_a T^a b_\mu + \xi_\mu \,, \qquad sb_\mu^\dagger = -i\omega_a b_\mu^\dagger T^a + \xi_\mu^\dagger \,. \tag{10}$$

In eq.(9) [...] means antisymmetrization with respect to the indices in the square bracket. ξ_μ and $\xi_\mu^\dagger$ are anti-commuting fields with FP-charge +1 whose BRST transformation is

$$s\xi_\mu = i\omega_a T^a \xi_\mu \,, \qquad s\xi_\mu^\dagger = -i\omega^a \xi_\mu^\dagger T^a \,. \tag{11}$$

[2]We work in the on-shell formalism, avoiding the introduction of the Nakanishi-Lautrup multiplier fields.

As anticipated, the ghost fields $\xi_\mu, \xi^\dagger_\mu$ pair under the BRST differential s with the fields $b_\mu, b^\dagger_\mu$.

The ghost fields $\xi_\mu, \xi^\dagger_\mu$ in the sector with FP-charge $+1$ are in correspondence with the antighost fields $\psi_\mu, \psi^\dagger_\mu$ in the sector with FP-charge -1. In order to preserve the correspondence between the sectors with negative and positive FP charge we need to introduce the field ζ and its conjugate $\zeta^\dagger$, with the following BRST transformation:

$$s\zeta = +i\omega_a T^a \zeta + \beta D^2 D^\rho b_\rho \,, \qquad s\zeta^\dagger = -i\omega_a \zeta^\dagger T^a - \beta (D^2(D^\rho b_\rho))^\dagger \,. \tag{12}$$

$\zeta, \zeta^\dagger$ have FP-charge -1. ζ is in the same representation as ϕ. β is a dimensionless parameter. We will comment on its rôle later on in Sect. 2.1.

The conjugation rules for the BRST differential s are the usual ones: for f fermion we have

$$sf^\dagger = -(sf)^\dagger \,, \tag{13}$$

while for b boson

$$sb^\dagger = (sb)^\dagger \,. \tag{14}$$

The action

$$\begin{aligned}
S &= Tr \int d^4x \, \Big[(D_\mu \phi + D^\nu b_{\nu\mu})^\dagger (D^\mu \phi + D_\rho b^{\rho\mu}) \\
&\qquad - \beta^2 (D^\rho b_\rho)^\dagger D^2 (D^{\rho'} b_{\rho'}) \\
&\qquad + \psi^\dagger_\mu (D^\mu h + D_\rho D^{[\rho} \xi^{\mu]}) - \psi_\mu (D^\mu h + D_\rho D^{[\rho} \xi^{\mu]})^\dagger \\
&\qquad - \beta \zeta^\dagger D^\rho \xi_\rho + \beta \zeta (D^\rho \xi_\rho)^\dagger \Big] \\
&= Tr \int d^4x \, \Big[(D_\mu \phi)^\dagger D^\mu \phi - b^\dagger_{\mu\nu} D^\mu D^\nu \phi - (D^\mu D^\nu \phi)^\dagger b_{\mu\nu} - b^\dagger_{\mu\nu} D^\mu D_\rho b^{\rho\nu} \\
&\qquad - \beta^2 (D^\rho b_\rho)^\dagger D^2 (D^{\rho'} b_{\rho'}) \\
&\qquad + \psi^\dagger_\mu (D^\mu h + D_\rho D^{[\rho} \xi^{\mu]}) - \psi_\mu (D^\mu h + D_\rho D^{[\rho} \xi^{\mu]})^\dagger \\
&\qquad - \beta \zeta^\dagger D^\rho \xi_\rho + \beta \zeta (D^\rho \xi_\rho)^\dagger \Big]
\end{aligned} \tag{15}$$

is BRST-invariant. s is nilpotent except on $\psi_\mu, \psi^\dagger_\mu$, on which it squares to zero modulo the equations of motion of $\psi^\dagger_\mu, \psi_\mu$:

$$s^2 \psi_\mu = \frac{\delta S}{\delta \psi^\dagger_\mu} \,, \qquad s^2 \psi^\dagger_\mu = \frac{\delta S}{\delta \psi_\mu} \,, \tag{16}$$

and on $\zeta, \zeta^\dagger$, on which we get

$$s^2\zeta = -D^2 \frac{\delta S}{\delta \zeta^\dagger}\,, \qquad s^2\zeta^\dagger = -\frac{\delta S}{\delta \zeta} \overleftarrow{D}^{2\dagger}\,. \tag{17}$$

We observe that s admits the following decomposition:

$$s = s_0 + s_1\,, \tag{18}$$

with

$$\begin{aligned} & s_0 A^a_\mu = (D_\mu \omega)^a\,, \qquad s_0 \omega^a = -\frac{1}{2} f^{abc} \omega^b \omega^c\,, \\ & s_0 \Phi = i\omega_a T^a \Phi\,, \qquad s_0 \Phi^\dagger = -i\omega_a \Phi^\dagger T^a\,, \end{aligned} \tag{19}$$

where Φ is a collective notation for $\{\phi, b_\mu, \psi_\mu, \xi_\mu, h, \zeta\}$, and

$$\begin{aligned} & s_1\phi = h\,, \qquad s_1 h = 0\,, \qquad s_1\phi^\dagger = h^\dagger\,, \qquad s_1 h^\dagger = 0\,, \\ & s_1 b_\mu = \xi_\mu\,, \qquad s_1\xi_\mu = 0\,, \qquad s_1 b^\dagger_\mu = \xi^\dagger_\mu\,, \qquad s_1\xi^\dagger_\mu = 0\,, \\ & s_1\psi_\mu = D_\mu\phi + D^\rho D_{[\rho} b_{\mu]}\,, \qquad s_1\psi^\dagger_\mu = -(D_\mu\phi + D^\rho D_{[\rho} b_{\mu]})^\dagger\,, \\ & s_1\zeta = \beta D^2 D^\rho b_\rho\,, \qquad s_1\zeta^\dagger = -\beta(D^2(D^\rho b_\rho))^\dagger\,, \\ & s_1 A^a_\mu = 0\,, \qquad s_1\omega_a = 0\,. \end{aligned} \tag{20}$$

s_0 in eq.(18) implements the gauge BRST transformations, s_1 the shift BRST transformations.

S in eq.(15) is simultaneously s_0- and s_1-invariant. Therefore it is also invariant under the following differential

$$s_\lambda = s_0 + \lambda s_1\,, \tag{21}$$

where λ is a commuting constant with FP-charge zero.

We now consider the following s_λ-invariant action

$$S_{embed} = -\frac{1}{4g^2} \int d^4x\, F^a_{\mu\nu} F^{\mu\nu\, a} + S\,, \tag{22}$$

where the field strength $F^a_{\mu\nu}$ is given by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu\,. \tag{23}$$

g is the coupling constant of the group G. The assignment of mass dimension of the fields is as follows: $A^a_\mu, \omega^a, \phi, \psi_\mu, \xi_\mu$ and the conjugates $\phi^\dagger, \psi^\dagger_\mu, \xi^\dagger_\mu$ have

dimension 1, ζ, h and the conjugate $\zeta^\dagger, h^\dagger$ have dimension 2 while b_μ and $b^\dagger_\mu$ have dimension zero.

S in eq.(15) has to be considered as a fixing functional for the shift symmetry associated with the BRST differential s_1. On the other hand the gauge invariance associated with the differential s_0 is still preserved by the full S_{embed} in eq.(22).

2.1 Spontaneous gauge symmetry breaking in the Higgs embedding model

The equations of motion for $\phi, \phi^\dagger$, derived from the action in eq.(22), admit the constant solution

$$\phi = \phi_v \, , \qquad \phi^\dagger = \phi^\dagger_v \, , \qquad \partial_\mu \phi_v = 0 \, , \tag{24}$$

with all other fields equal to zero. The shift

$$\phi \rightarrow \phi_v + \phi \, , \qquad \phi^\dagger \rightarrow \phi^\dagger_v + \phi^\dagger \tag{25}$$

yields for the action S_{embed}

$$\begin{aligned} S_{embed} &= -\frac{1}{4g^2} \int d^4x \, F^a_{\mu\nu} F_{\mu\nu a} + S \\ &\quad + \int d^4x \, A_{\mu a} M^{ab} A^\mu_b \\ &\quad + Tr \int d^4x \, \left(i A_{\mu a} \phi^\dagger_v T^a D^\mu \phi + h.c. \right) \\ &\quad + Tr \int d^4x \, \left(\frac{i}{2} b^{\mu\nu\dagger} F^a_{\mu\nu} T^a \phi_v + h.c. \right) \end{aligned} \tag{26}$$

where

$$M^{ab} = Tr[\phi^\dagger_v T^a T^b \phi_v] \, . \tag{27}$$

The term in the second line in eq.(26) is a mass term for the non-Abelian gauge fields of the same kind as in the usual Higgs mechanism.

Some comments are in order here. Due to the shift symmetry s_1 it is possible to introduce the term $(D_\mu \phi)^\dagger (D^\mu \phi)$ only via the functional in the

first line of eq.(15). The latter provides the fixing condition for the shift symmetry of the model. The needed vector antighost field ψ_μ requires the introduction of the vector ghost field ξ_μ, pairing with the vector field b^μ into a s_1-doublet. The antighost field ζ is associated with the s_1-partner h of ϕ and is needed to generate the term

$$-\beta^2 (Db)^\dagger D^2 (Db)$$

in the second line of eq.(15).

In the absence of such a term (i.e. at $\beta = 0$) the propagator of b_μ does not exist. We remark that, as a consequence of the shift symmetry s_1, no quartic potential for the field ϕ is allowed.

2.2 BV formulation of the Higgs model embedding

Since the BRST differential s squares to zero only modulo the equations of motion of $\psi_\mu, \psi_\mu^\dagger$ and of $\zeta, \zeta^\dagger$, the Batalin-Vilkovisky (BV) formalism must be used to derive the solution $\Gamma^{(0)}$ to the classical ST identities of the model [15].

$\Gamma^{(0)}$ contains terms quadratic in the antifields $\psi_\mu^{*\dagger}, \psi_\mu^*$ and $\zeta^{*\dagger}, \zeta^*$ and is given by

$$\begin{aligned}
\Gamma^{(0)} \quad = \quad & S_{embed} + \int d^4x \, \Big[\frac{\alpha}{2} \Big(\partial A^a + \frac{1}{\alpha} Tr(i\phi_v^\dagger T^a \phi - i\phi^\dagger T^a \phi_v) \Big)^2 \\
& - \alpha \bar{\omega}^a s \Big(\partial A^a + \frac{1}{\alpha} Tr(i\phi_v^\dagger T^a \phi - i\phi^\dagger T^a \phi_v) \Big) \Big] \\
& + \int d^4x \, A_\mu^{a*} (D_\mu \omega)^a - \int d^4x \, \omega^{a*} \frac{1}{2} f^{abc} \omega^b \omega^c \\
& + Tr \int d^4x \, \chi^* s\chi + Tr \int d^4x \, \chi^{*\dagger} s \chi^\dagger \\
& + Tr \int d^4x \, \left(-\psi_\mu^{*\dagger} \psi^{*\mu} + \zeta^* D^2 \zeta^{\dagger *} \right) . \qquad (28)
\end{aligned}$$

where S_{embed} is given in eq.(26), χ is a collective notation for $\{\phi, h, b_\mu, \xi_\mu, \psi_\mu, \zeta\}$ and χ^* stands for the antifields $\{\phi^*, h^*, b_\mu^*, \xi_\mu^*, \psi_\mu^*, \zeta^*\}$. We have chosen a R_α-gauge by setting

$$s\bar{\omega}^a = \partial A^a + \frac{1}{\alpha} Tr(i\phi_v^\dagger T^a \phi - i\phi^\dagger T^a \phi_v) \, . \qquad (29)$$

$\bar{\omega}^a$ is the antighost for the gauge ghost field ω^a. The R.H.S. of eq.(29) is linear in the quantum fields, therefore we do not need an antifield for $\bar{\omega}^a$.

$\Gamma^{(0)}$ satisfies the following Slavnov-Taylor (ST) identities:

$$\begin{aligned}\mathcal{S}(\Gamma^{(0)}) &= \int d^4x \left(\frac{\delta\Gamma^{(0)}}{\delta A^{a*}_\mu}\frac{\delta\Gamma^{(0)}}{\delta A^\mu_a} + \frac{\delta\Gamma^{(0)}}{\delta\omega^{a*}}\frac{\delta\Gamma^{(0)}}{\delta\omega_a} + Tr\frac{\delta\Gamma^{(0)}}{\delta\chi^*}\frac{\delta\Gamma^{(0)}}{\delta\chi} + Tr\frac{\delta\Gamma^{(0)}}{\delta\chi^{*\dagger}}\frac{\delta\Gamma^{(0)}}{\delta\chi^\dagger}\right.\\ &\left. + \left(\partial A^a + \frac{1}{\alpha}Tr(i\phi_v^\dagger T^a\phi - i\phi^\dagger T^a\phi_v)\right)\frac{\delta\Gamma^{(0)}}{\delta\bar{\omega}_a}\right) = 0\,. \qquad (30)\end{aligned}$$

The linearized classical ST operator $\mathcal{S}_0$ is given by

$$\begin{aligned}\mathcal{S}_0 &= \int d^4x \left(\frac{\delta\Gamma^{(0)}}{\delta A^{a*}_\mu}\frac{\delta}{\delta A^\mu_a} + \frac{\delta\Gamma^{(0)}}{\delta A^\mu_a}\frac{\delta}{\delta A^{a*}_\mu} + \frac{\delta\Gamma^{(0)}}{\delta\omega^{a*}}\frac{\delta}{\delta\omega_a} + \frac{\delta\Gamma^{(0)}}{\delta\omega_a}\frac{\delta}{\delta\omega^{a*}}\right.\\ &+ Tr\frac{\delta\Gamma^{(0)}}{\delta\chi^*}\frac{\delta}{\delta\chi} + Tr\frac{\delta\Gamma^{(0)}}{\delta\chi}\frac{\delta}{\delta\chi^*} + Tr\frac{\delta\Gamma^{(0)}}{\delta\chi^{*\dagger}}\frac{\delta}{\delta\chi^\dagger} + Tr\frac{\delta\Gamma^{(0)}}{\delta\chi^\dagger}\frac{\delta}{\delta\chi^{*\dagger}}\\ &\left. + \left(\partial A^a + \frac{1}{\alpha}Tr(i\phi_v^\dagger T^a\phi - i\phi^\dagger T^a\phi_v)\right)\frac{\delta}{\delta\bar{\omega}_a}\right)\,. \qquad (31)\end{aligned}$$

$\mathcal{S}_0$ is nilpotent on the space of functionals obeying the ghost equation

$$\frac{\delta\Gamma^{(0)}}{\delta\bar{\omega}^a} = -\alpha\partial^\mu\frac{\delta\Gamma^{(0)}}{\delta A^*_{\mu a}} - Tr\left[i\phi_v^\dagger T^a\frac{\delta\Gamma^{(0)}}{\delta\phi^*} - i\frac{\delta\Gamma^{(0)}}{\delta\phi^{\dagger *}}T^a\phi_v\right]\,. \qquad (32)$$

In the following subsection we will apply the embedding formalism for the Higgs model to the example of the gauge group $SU(2)$ and discuss the UV and IR properties of the propagators after the shift $\phi \to \phi + \phi_v$.

2.3 The case of the $SU(2)$ gauge group

We denote the Pauli matrices by T^a, $a = 1, 2, 3$. They obey the $SU(2)$ commutation relation

$$[T^a, T^b] = 2i\epsilon^{abc}T^c\,. \qquad (33)$$

By comparison with eq.(1) we get $f^{abc} = 2\epsilon^{abc}$. The field ϕ is a complex doublet which we parameterize as follows:

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} i\phi_1 + \phi_2 \\ \phi_0 + v - i\phi_3\end{pmatrix}\,. \qquad (34)$$

In a similar fashion we write for b_μ

$$b_\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} ib_{1\mu} + b_{2\mu} \\ b_{0\mu} - ib_{3\mu} \end{pmatrix} . \tag{35}$$

The covariant derivative acting on ϕ and on all other fields in the same representation is

$$D_\mu \phi(x) = \partial_\mu \phi(x) - igA^a_\mu T^a \phi(x) \, , \tag{36}$$

where we have restored the coupling constant g in front of A^a_μ. We choose a R_α-gauge-fixing condition according to

$$s\bar{\omega}^a = \partial A^a - \frac{gv}{\alpha}\phi^a \, . \tag{37}$$

Now we can compute the quadratic part in the bosonic fields in the classical action in eq.(28):

$$\begin{aligned} \Sigma_2 \quad = \quad & \int d^4x \Big(+ \frac{1}{2} \sum_{a=1}^{3} A^a_\mu ((\Box + (gv)^2) g^{\mu\nu} - (1+\alpha)\partial^\mu \partial^\nu) A^a_\nu \\ & + \frac{1}{2} \partial_\mu \phi_0 \partial^\mu \phi_0 + \sum_{a=1}^{3} \Big(\frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a + \frac{1}{2} \frac{(gv)^2}{\alpha} \phi_a^2 \Big) \\ & + \frac{1}{2} b^\mu_0 (\Box^2 g_{\mu\nu} - (1-\beta^2) \Box \partial_\mu \partial_\nu) b^\nu_0 \\ & + \frac{1}{2} \sum_{a=1}^{3} b^\mu_a (\Box^2 g_{\mu\nu} - (1-\beta^2) \Box \partial_\mu \partial_\nu) b^\nu_a \\ & + \frac{gv}{2} \sum_{a=1}^{3} \partial^{[\mu} b^{\nu]}_a \hat{F}^a_{\mu\nu} \Big) \, , \end{aligned} \tag{38}$$

where $\hat{F}^a_{\mu\nu}$ denotes the abelianization of the field strength $F^a_{\mu\nu}$:

$$\hat{F}^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \, . \tag{39}$$

Notice the appearance of the off-diagonal terms in the last line of the eq.(38).

We parameterize the antighosts ψ_μ and ζ and the ghosts ξ_μ, h as follows:

$$\begin{aligned} \psi_\mu = \tfrac{1}{\sqrt{2}} \begin{pmatrix} -\psi_{1\mu} + i\psi_{2\mu} \\ \psi_{3\mu} + i\psi_{0\mu} \end{pmatrix} , \qquad \zeta = \tfrac{1}{\sqrt{2}} \begin{pmatrix} -\zeta_1 + i\zeta_2 \\ \zeta_3 + i\zeta_0 \end{pmatrix} , \\ \xi_\mu = \tfrac{1}{\sqrt{2}} \begin{pmatrix} i\xi_{1\mu} + \xi_{2\mu} \\ \xi_{0\mu} - i\xi_{3\mu} \end{pmatrix} , \qquad h = \tfrac{1}{\sqrt{2}} \begin{pmatrix} ih_1 + h_2 \\ h_0 - ih_3 \end{pmatrix} . \end{aligned} \tag{40}$$

Then the quadratic part in the ghost-antighost fields in eq.(28) is

$$\begin{aligned}\Sigma_{gh,2} &= \int d^4x \Bigg[i\Bigg(-\psi_{0\mu}\Big((\Box g^{\mu\rho}-\partial^\mu\partial^\rho)\xi_{0\rho}+\partial^\mu h_0\Big) + \beta\zeta_0\partial\xi_0 \\ &\quad - \sum_{a=1}^{3}\Big(\psi_{a\mu}\Big((\Box g^{\mu\rho}-\partial^\mu\partial^\rho)\xi_{a\rho}+\partial^\mu h_a\Big) - \beta\zeta_a\partial\xi_a\Big)\Bigg) \\ &\quad + \sum_{a=1}^{3}\Big(-\alpha\bar\omega_a\Box\omega_a + (gv)^2\bar\omega_a\omega_a + gv\bar\omega_a h_a\Big)\Bigg] . \end{aligned} \tag{41}$$

By inverting eq.(38) and eq.(41) we obtain the UV δ and IR ρ indices [24]-[27] of the fields, which are given by

$$\begin{aligned} &\delta(\phi_0)=\rho(\phi_0)=1\,, \quad \delta(b_{0\mu})=\rho(b_{0\mu})=0\,, \\ &\delta(\phi_a)=1\,,\ \rho(\phi_a)=2\,, \quad a=1,2,3\,, \end{aligned} \tag{42}$$

and by

$$\begin{aligned} &\delta(\psi_{i\mu})=\rho(\psi_{i\mu})=1\,, \quad \delta(\xi_{i\mu})=\rho(\xi_{i\mu})=1\,, \\ &\delta(h_i)=\rho(h_i)=2\,, \quad \delta(\zeta_i)=\rho(\zeta_i)=2\,, \quad i=0,1,2,3\,, \\ &\delta(\bar\omega_a)=\delta(\omega_a)=1\,, \quad \rho(\bar\omega_a)=\rho(\omega_a)=2\,, \quad a=1,2,3\,. \end{aligned} \tag{43}$$

In the sector $A^a_\mu - b^c_\nu$ we find the following propagators [3]

$$\Delta_{b^c_\mu(-p)b^d_\nu(p)} = \delta^{cd}\frac{-i}{(p^2)^2F(p^2)}\left(g_{\mu\nu}+\frac{F(p^2)-\beta^2}{\beta^2}\frac{p_\mu p_\nu}{p^2}\right)\,, \tag{44}$$

where we have set

$$F(p^2) = 1-\frac{g^2v^2}{-p^2+(gv)^2}\,, \tag{45}$$

and

$$\Delta_{A^a_\mu(-p)b^d_\nu(p)} = \delta^{ad}\frac{igv}{(p^2)^3}\left(-p^2g_{\mu\nu}+p_\mu p_\nu\right)\,, \tag{46}$$

$$\Delta_{A^a_\mu(-p)A^b_\nu(p)} = \Delta_{A'^a_\mu(-p)A'^b_\nu(p)} + \delta^{ab}\frac{i(gv)^2}{(p^2-(gv)^2)^2}\frac{1}{p^2F(p^2)}(-p^2g_{\mu\nu}+p_\mu p_\nu) \tag{47}$$

[3] for notational convenience we omit to write the $i\epsilon$ factor.

with

$$\Delta_{A'^a_\mu(-p)A'^b_\nu(p)} = \delta^{ab}\frac{i}{p^2-(gv)^2}\left(g_{\mu\nu} - \frac{(1+\alpha)}{\alpha p^2+(gv)^2}p_\mu p_\nu\right) . \tag{48}$$

Since $F(p^2)$ tends to 1 for large p the UV degrees are

$$\delta(A^a_\mu) = 1\,, \qquad \delta(b^a_\nu) = 0\,, \tag{49}$$

compatible with the power-counting renormalizability criteria. In the infrared regime the behavior of the propagators in eqs.(44)-(47) yields the assignments

$$\rho(A^a_\mu) = 1\,, \qquad \rho(b^b_\nu) = -1\,, \tag{50}$$

which then lead to a bad IR power-counting (the classical action contains vertices with IR degree $<$ 4). This destroys the renormalizability of the model by power-counting.

Let us comment on the results obtained so far. We have discussed the embedding of the Higgs model into a larger theory which is invariant under the BRST differential s in eq.(18). No quartic potential for the field ϕ is any longer allowed, since it is forbidden by the shift symmetry s_1. The analysis of the propagators and the interaction vertices shows that the UV conditions for power-counting renormalizability are fulfilled, while the IR ones are not. This is due to a bad IR asymptotic behaviour of the propagators in the sector spanned by the fields A^a_μ and b^b_ν. As a consequence, the Higgs model embedding is not power-counting renormalizable.

We conclude that the embedding procedure in the presence of the Higgs mechanism fails to produce the sought-for power-counting renormalizable embedding of massive Yang-Mills theory.

Nevertheless, despite its failure the model just analyzed provides a clue towards the correct embedding of massive Yang-Mills theory. In this connection we would like to comment on the shift symmetry fixing terms given by the s_1-variations of $\psi_\mu, \psi^\dagger_\mu$ and $\zeta, \zeta^\dagger$ in eq.(20).

According to eqs.(16) and (17) s squares to zero only modulo the equations of motion of $\psi_\mu, \psi^\dagger_\mu$ and $\zeta, \zeta^\dagger$. Both eqs.(16) and (17) can be dealt with by the BV formalism. Nevertheless there exists an important difference between eq.(16) and eq.(17): the first one can be implemented in an off-shell

formulation by introducing suitable Nakanishi-Lautrup multiplier fields for $\psi_\mu, \psi^\dagger_\mu$, the second one does not admit such an off-shell formulation.

This suggests that there should exist another embedding procedure where the fields $\zeta, \zeta^\dagger$ (and consequently $\phi, \phi^\dagger$ and $h, h^\dagger$) do not appear. Such a minimal embedding model can actually be constructed. It turns out that it naturally solves the IR problems affecting the Higgs model embedding. Moreover, we will prove that it gives rise to a true power-counting renormalizable model for massive non-Abelian gauge bosons without spontaneous gauge symmetry breaking. As a consequence of the embedding BRST symmetry, the physical observables are given by gauge-invariant quantities constructed only from the field strength and its covariant derivatives.

The analysis of the minimal embedding formalism will be carried out in the next section.

3 Minimal embedding

In the minimal embedding formalism we drop the fields $\phi, \phi^\dagger$ and their ghost partners $h, h^\dagger$ together with the antighost fields $\zeta, \zeta^\dagger$.

The fixing action S is now given by

$$\begin{aligned} S &= Tr \int d^4x \, K^\dagger_\mu K^\mu \\ & \quad + Tr \int d^4x (\psi^\dagger_\mu s K^\mu + (sK^\mu)^\dagger \psi_\mu) \, , \end{aligned} \tag{51}$$

where

$$K_\mu \equiv D_\mu(Db) + \lambda D^2 b_\mu + \gamma \frac{m^2}{2} b_\mu + D_\mu \underline{M} \, . \tag{52}$$

In the above equation we have set

$$\underline{M} = \frac{1}{\sqrt{2}} \underline{m} \, . \tag{53}$$

with $\underline{M}, \underline{m}$ in the same representation as b_μ. For instance, in the case of the $SU(2)$ group we get

$$\underline{M} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ m \end{pmatrix} \, . \tag{54}$$

m is a parameter with the dimension of a mass.

λ, γ are real parameters. We will comment later on on their physical significance. The BRST transformations of b^μ and their ghost partners ξ^μ are the same as in eqs.(10) and (11), while

$$s\psi_\mu = K_\mu \, , \qquad s\psi^\dagger_\mu = -K^\dagger_\mu \, . \tag{55}$$

The explicit form of sK_μ is

$$\begin{aligned} sK_\mu &= i\omega_a T^a (D_\mu(Db) + \lambda D^2 b_\mu + \gamma \frac{m^2}{2} b_\mu) \\ &\quad + D_\mu(D\xi) + \lambda D^2 \xi_\mu + \gamma \frac{m^2}{2} \xi_\mu \\ &\quad - i(D_\mu \omega)_a T^a \underline{M} \, . \end{aligned} \tag{56}$$

We notice that gauge invariance is explicitly broken by the last term in eq.(56). In this model there is no spontaneous gauge symmetry breaking.

S in eq.(51) is s-invariant. s squares to zero modulo the equations of motion of $\psi^\mu, \psi^{\mu\dagger}$:

$$s^2 \psi_\mu = \frac{\delta S}{\delta \psi^{\mu\dagger}} \, , \qquad s^2 \psi^\dagger_\mu = \frac{\delta S}{\delta \psi^\mu} \, . \tag{57}$$

Due to eq.(57) an off-shell formulation of the model exists, based on the introduction of the Nakanishi-Lautrup multiplier fields which form BRST doublets together with $\psi_\mu, \psi^\dagger_\mu$. Hence S turns out to be a true fixing functional for the symmetry s.

In what follows we choose to work in the on-shell formalism and we shall use the BV method in order to derive the relevant ST identities of the model.

The embedding action reads

$$S_{\rm embed} = -\frac{1}{4g^2} \int d^4x \, F^a_{\mu\nu} F^{\mu\nu a} + S \, . \tag{58}$$

This action is s-invariant. Notice that we have not yet chosen the gauge-fixing function for the vector fields A^a_μ.

3.1 Analysis of the UV and IR properties of the model

In this subsection we analyze the UV and IR properties of the minimal embedding model by deriving the UV and IR degrees of the fields and by

checking that the interaction vertices obey the conditions for power-counting renormalizability [24, 25, 26, 27].

For the sake of definiteness we specialize from now on to the case of the gauge group $SU(2)$. Nevertheless the analysis can be straightforwardly extended to other gauge groups of interest, including those involving Abelian factors like $SU(2) \times U(1)$.

We use the same conventions about normalizations and covariant derivatives as in Sect. 2.3.

3.1.1 The bosonic sector

The quadratic part of $S_{\rm embed}$ in eq.(58) is given in the bosonic sector by

$$
\begin{aligned}
S_{\rm embed,II} = & \int d^4x \Big(- \frac{1}{4} \sum_{a=1}^{3} \hat{F}^a_{\mu\nu} \hat{F}^{\mu\nu\, a} \\
& + \frac{1}{2} (\partial_\mu (\partial b_0) + \lambda \Box b_{0\mu} + \frac{\gamma m^2}{2} b_{0\mu})^2 \\
& + \sum_{a=1}^{3} \frac{1}{2} (\partial_\mu (\partial b_a) + \lambda \Box b_{a\mu} + \frac{\gamma m^2}{2} b_{a\mu} - g m A_{a\mu})^2 \Big) \, ,
\end{aligned} \tag{59}
$$

where $\hat{F}^a_{\mu\nu}$ denotes the Abelianization of the field strength $F^a_{\mu\nu}$:

$$
\hat{F}^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \, . \tag{60}
$$

We see from eq.(59) that for non-exceptional values of the parameters λ and γ the UV and IR degrees of $b_{0\mu}$ are

$$
\delta(b_{0\mu}) = 0 \, , \qquad \rho(b_{0\mu}) = 2 \, . \tag{61}
$$

The quadratic part of $S_{\rm embed}$ in the sector $A_{\mu a} - b_{\nu b}$ is

$$
\begin{aligned}
S_{\rm embed,II,A-b} = & \int d^4x \Big[\sum_{a=1}^{3} \frac{1}{2} b_{a\mu} \Big[\Big(\lambda^2 \Box^2 + \lambda \gamma m^2 \Box + \Big(\frac{\gamma m^2}{2} \Big)^2 \Big) g^{\mu\nu} \\
& + ((1+2\lambda)\Box + \gamma m^2) \partial^\mu \partial^\nu \Big] b_{a\nu} \\
& + \sum_{a=1}^{3} \frac{1}{2} A^a_\mu \Big((\Box + g^2 m^2) g^{\mu\nu} - \partial^\mu \partial^\nu \Big) A^a_\nu
\end{aligned}
$$

$$+\sum_{a=1}^{3}\Big(gm\,\partial A^a\partial b^a - gmA^{\mu a}\Big(\lambda\Box+\frac{\gamma m^2}{2}\Big)b^a_\mu\Big)\Big]\,. \tag{62}$$

The mixed term $\partial A_a \partial b_a$ can be removed by making use of the following R_α -gauge-fixing:

$$\frac{\alpha}{2}\Big(\partial A_a - \frac{gm}{\alpha}\partial b_a\Big)^2 \tag{63}$$

Then eq.(62) becomes

$$\begin{aligned} S_{\rm embed,II,A-b} \;&=\; \int d^4x\,\Big[\sum_{a=1}^{3}\frac{1}{2}b_{a\mu}\Big[\Big(\lambda^2\Box^2+\lambda\gamma m^2\Box+\Big(\frac{\gamma m^2}{2}\Big)^2\Big)g^{\mu\nu} \\ &\quad + ((1+2\lambda)\Box+\gamma m^2-\frac{(gm)^2}{\alpha})\partial^\mu\partial^\nu\Big]b_{a\nu} \\ &\quad + \frac{1}{2}\sum_{a=1}^{3}A_{\mu a}\Big((\Box+g^2m^2)g^{\mu\nu}-(1+\alpha)\partial^\mu\partial^\nu\Big)A_{a\nu} \\ &\quad - \sum_{a=1}^{3}gmA_{a\mu}\Big(\lambda\Box+\frac{\gamma m^2}{2}\Big)b^\mu_a\Big]\,. \end{aligned} \tag{64}$$

By inverting the two-point function matrix in eq.(64) we obtain the following assignments of the UV and IR indices:

$$\begin{aligned} &\delta(A_{a\mu})=1\,, \qquad \rho(A_{a\mu})=1\,, \\ &\delta(b_{a\mu})=0\,, \qquad \rho(b_{a\mu})=1\,. \end{aligned} \tag{65}$$

3.1.2 The ghost sector

The gauge-fixing choice in eq.(63) implies that

$$s\bar\omega^a = \partial A^a - \frac{gm}{\alpha}\partial b^a\,. \tag{66}$$

Therefore the $\bar\omega^a$-dependent part of the classical action reads

$$-\int d^4x\,\alpha\bar\omega^a\Big(\partial_\mu(D^\mu\omega)^a - \frac{gm}{\alpha}\partial\xi^a - \frac{g^2m}{\alpha}(\partial^\mu(\omega^a b_{0\mu}) - \epsilon^{abc}\partial^\mu(\omega_b b_{c\mu}))\Big)\,. \tag{67}$$

The quadratic part in the ghost sector is then given by

$$
\begin{aligned}
\Sigma_{\rm II,ghost} \;=\; & \int d^4x \Big(-i\sum_{a=1}^{3}\psi_{a\mu}\Big(\partial^\mu(\partial\xi_a)+\lambda\Box\xi_a^\mu+\frac{1}{2}\gamma m^2\xi_a^\mu-gm\partial^\mu\omega_a\Big) \\
& -i\psi_{0\mu}\Big(\partial^\mu(\partial\xi_0)+\lambda\Box\xi_0^\mu+\frac{1}{2}\gamma m^2\xi_0^\mu\Big) \\
& -\alpha\bar\omega_a\Big(\Box\omega_a-\frac{gm}{\alpha}\partial\xi_a\Big)\Big)\,. \qquad (68)
\end{aligned}
$$

The inversion of the ghost two-point function matrix in eq.(68) yields the following assignments of the UV and IR indices:

$$
\begin{aligned}
&\delta(\psi_{0\mu})=\delta(\psi_{a\mu})=\delta(\xi_{0\mu})=\delta(\xi_{a\mu})=1\,, \qquad \delta(\bar\omega^a)=\delta(\omega^a)=1\,,\\
&\rho(\psi_{0\mu})=\rho(\psi_{a\mu})=\rho(\xi_{0\mu})=\rho(\xi_{a\mu})=2\,, \qquad \rho(\bar\omega^a)=\rho(\omega^a)=1\,. \qquad (69)
\end{aligned}
$$

A table summarizing the UV and IR degrees of all the fields of the model is reported in Appendix C.

By using the IR and UV assignments in eq.(61), (65) and (69) one can verify that all the interaction vertices in $S_{\rm embed}$ in eq.(58) have UV degree ≤ 4 and IR degree ≥ 4. Hence they satisfy the power-counting conditions [24, 25, 26, 27] which are sufficient to ensure the power-counting renormalizability of the model. The minimal embedding procedure thus naturally solves the IR difficulties discussed in Sect. 2.3 for the Higgs model embedding.

3.2 Classical Slavnov-Taylor identities

Since in the on-shell formalism the BRST differential s squares to zero only modulo the equations of motions of $\psi^\dagger_\mu, \psi_\mu$, we use the BV method to obtain the relevant ST identities of the model.

The classical action $\Gamma^{(0)}$ fulfilling these ST identities includes a term quadratic in the antifields $\psi^{\dagger *}_\mu, \psi^*_\mu$. $\Gamma^{(0)}$ is given by

$$
\begin{aligned}
\Gamma^{(0)} \;=\; & \int d^4x \Big(-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}+\frac{\alpha}{2}\Big(\partial A^a-\frac{gm}{\alpha}\partial b^a)^2 \\
& +Tr\Big(K^\dagger_\mu K^\mu+\psi^\dagger_\mu sK^\mu+(sK_\mu)^\dagger\psi^\mu\Big) \\
& -\alpha\bar\omega^a\Big(\partial^\mu(D_\mu\omega)^a-\frac{gm}{\alpha}\partial\xi^a-\frac{g^2m}{\alpha}\partial^\mu(\omega^a b_{0\mu}-\epsilon^{abc}\omega^b b^c_\mu)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+A^{a*}_\mu (D^\mu \omega)^a - \omega^{a*}\frac{1}{2} g f^{abc}\omega^b\omega^c \\
&+Tr\Big(b^*_\mu\, sb^\mu + b^{\dagger *}_\mu\, sb^{\dagger\mu} + \xi^*_\mu\, s\xi^\mu + \xi^{\dagger *}_\mu\, s\xi^{\dagger\mu}\Big) \\
&+Tr\Big(\psi^*_\mu K^\mu - \psi^{\dagger *}_\mu K^{\mu\dagger} - \psi^{\dagger *}_\mu \psi^{\mu *}\Big)\Big)\,. \qquad (70)
\end{aligned}
$$

No differential operators enter in the last term in eq.(70), unlike in the case of the last monomial in eq.(28).

$\Gamma^{(0)}$ fulfills the following ST identities

$$
\begin{aligned}
\mathcal{S}(\Gamma^{(0)}) \;=\; \int d^4x \, \Big(& \frac{\delta\Gamma^{(0)}}{\delta A^{a*}_\mu}\frac{\delta\Gamma^{(0)}}{\delta A^a_\mu} + \frac{\delta\Gamma^{(0)}}{\delta\omega^{a*}}\frac{\delta\Gamma^{(0)}}{\delta\omega^a} \\
& + Tr\frac{\delta\Gamma^{(0)}}{\delta\chi^*}\frac{\delta\Gamma^{(0)}}{\delta\chi} \\
& + \Big(\partial A^a - \frac{gm}{\alpha}\partial b^a\Big)\frac{\delta\Gamma^{(0)}}{\delta\bar\omega^a}\Big) = 0\,. \qquad (71)
\end{aligned}
$$

In the above equation χ stands for $\chi = \{b_\mu, b^\dagger_\mu, \xi_\mu, \xi^\dagger_\mu, \psi_\mu, \psi^\dagger_\mu\}$ and χ^* for $\chi^* = \{b^*_\mu, b^{\dagger *}_\mu, \xi^*_\mu, \xi^{\dagger *}_\mu, \psi^*_\mu, \psi^{\dagger *}_\mu\}$. Notice that we do not introduce an antifield for $\bar\omega^a$, since its BRST transformation is linear in the quantum fields.

The classical linearized ST operator $\mathcal{S}_0$ is given by

$$
\begin{aligned}
\mathcal{S}_0 \;=\; \int d^4x \, \Big(& \frac{\delta\Gamma^{(0)}}{\delta A^{a*}_\mu}\frac{\delta}{\delta A^a_\mu} + \frac{\delta\Gamma^{(0)}}{\delta A^a_\mu}\frac{\delta}{\delta A^{a*}_\mu} + \frac{\delta\Gamma^{(0)}}{\delta\omega^{a*}}\frac{\delta}{\delta\omega^a} + \frac{\delta\Gamma^{(0)}}{\delta\omega^a}\frac{\delta}{\delta\omega^{a*}} \\
& + Tr\frac{\delta\Gamma^{(0)}}{\delta\chi^*}\frac{\delta}{\delta\chi} + Tr\frac{\delta\Gamma^{(0)}}{\delta\chi}\frac{\delta}{\delta\chi^*} + \Big(\partial A^a - \frac{gm}{\alpha}\partial b^a\Big)\frac{\delta}{\delta\bar\omega^a}\Big)\,. \qquad (72)
\end{aligned}
$$

$\mathcal{S}_0$ is nilpotent on the space of functionals satisfying the ghost equation

$$
\frac{\delta\Gamma^{(0)}}{\delta\bar\omega_a} = -\alpha\partial^\mu\frac{\delta\Gamma^{(0)}}{\delta A^{a*}_\mu} - \frac{gm}{\alpha}\partial^\mu\frac{\delta\Gamma^{(0)}}{\delta b^{a*}_\mu}\,. \qquad (73)
$$

3.3 Analysis of the cohomology of $\mathcal{S}_0$

All the fields and antifields of the model with the exception of A^a_μ, ω^a and their antifields and $\bar\omega^a$ pair into $\mathcal{S}_0$-doublets. Hence the cohomology $H(\mathcal{S}_0)$ in the space of functionals fulfilling the ghost equation

$$
\frac{\delta X}{\delta\bar\omega^a} = -\alpha\partial^\mu\frac{\delta X}{\delta A^{a*}_\mu} - \frac{gm}{\alpha}\partial^\mu\frac{\delta X}{\delta b^{a*}_\mu} \qquad (74)
$$

is isomorphic to the cohomology of the restriction $\mathcal{S}'_0$ of $\mathcal{S}_0$ to the space of functionals spanned by A^a_μ, ω^a and their antifields and by $\bar{\omega}^a$, which obey eq.(74) [20, 16, 21].

The cohomology of such a restriction is known to be isomorphic [20, 16] to the cohomology $H(s_0)$ of the pure Yang-Mills differential s_0:

$$s_0 A^a_\mu = (D_\mu \omega)^a \,, \qquad s_0 \omega^a = -\frac{1}{2} g f^{abc} \omega^b \omega^c \,. \tag{75}$$

In the FP-neutral sector (physical observables) $H(s_0)$ is given by the gauge-invariant quantities constructed only from the field strength $F^a_{\mu\nu}$ and its covariant derivatives. These are the physical observables of the minimal embedding model.

In the sector with FP-charge +1 (anomalies) the cohomology of $\mathcal{S}'_0$ in the relevant functional space fulfilling eq.(74) is empty [20, 16]. Hence there are no candidate anomalies for the quantum extension of the ST identities in eq.(71).

Together with power-counting renormalizability, guaranteeing the validity of the Quantum Action Principle [16], this implies that the quantum ST identities

$$\mathcal{S}(\mathrm{I\!\Gamma}) = 0 \tag{76}$$

can be restored order by order in the loop expansion by a suitable choice of finite counterterms, yielding the symmetric quantum effective action

$$\mathrm{I\!\Gamma} = \sum_{j=0}^{\infty} \mathrm{I\!\Gamma}^{(j)} \tag{77}$$

which fulfills eq.(76). In eq.(77) $\mathrm{I\!\Gamma}^{(j)}$ stands for the coefficient of order j in the loop expansion of $\mathrm{I\!\Gamma}$. $\mathrm{I\!\Gamma}^{(0)}$ coincides with $\Gamma^{(0)}$ in eq.(71).

3.4 The spectrum of the classical theory

As a consequence of the cohomological analysis developed in the previous subsection, we know that the physical observables of the minimal embedding model are given by gauge-invariant quantities constructed only from the field strength $F^a_{\mu\nu}$ and its covariant derivatives.

We wish to complete the analysis of the physical content of the theory by computing the physical spectrum of the classical action in eq.(70). This will also clarify the meaning of the parameters λ and γ in eq.(52). For this purpose the study of the relevant asymptotic states is needed.

We first analyze the sector spanned by A^a_μ and b^c_ν. From now on we choose to work in the Feynman gauge $\alpha = -1$. The quadratic part in A^a_μ, b^c_ν in eq.(64) is given in the momentum space by

$$\begin{aligned} S_{\mathrm{embed,II,A-b}} \;=\; & \int d^4p \Bigg[\sum_{a=1}^{3} \frac{1}{2} b_{a\mu}(-p) \bigg[\Big(\lambda p^2 - \frac{\gamma m^2}{2} \Big)^2 g^{\mu\nu} \\ & + \Big((1+2\lambda)p^2 - \gamma m^2 - (gm)^2 \Big) p^\mu p^\nu \bigg] b_{a\nu}(p) \\ & + \sum_{a=1}^{3} \frac{1}{2} A_{a\mu}(-p)(-p^2 + g^2 m^2) A^\mu_a(p) \\ & + \sum_{a=1}^{3} A_{a\mu}(-p) gm \Big(\lambda p^2 - \frac{\gamma m^2}{2} \Big) b^\mu_a(p) \Bigg] . \end{aligned} \tag{78}$$

The asymptotic states are the elements of the kernel of the two-point function matrix $\underline{\Sigma} = \Sigma_{(\mu,a,L;\ \mu',a',L')}$ computed from eq.(78):

$$\Sigma_{(\mu,a,L;\ \mu',a',L')} = \frac{\delta^2 S_{\mathrm{embed,II,A-b}}}{\delta\varphi^{a,L}_\mu(-p)\delta\varphi^{a',L'}_{\mu'}(p)} \tag{79}$$

where we have denoted by $\varphi^{a,L}_\mu$ the column vector with components

$$\varphi^{a,1}_\mu = A^a_\mu \,, \qquad \varphi^{a,2}_\mu = b^a_\mu \,.$$

$\underline{\Sigma}$ is diagonal in the aa'-space.

The values of the masses in the spectrum are given by the zeroes of $\Sigma = \det \underline{\Sigma}$. For arbitrary values of λ and γ Σ vanishes at

$$p^2 = 0 \,, \qquad p^2 = \frac{\gamma m^2}{2\lambda} \,, \qquad p^2 = \frac{(2g^2+\gamma)m^2}{2(1+\lambda)} . \tag{80}$$

The requirement that the spectrum includes the point $p^2 = g^2m^2$ imposes a constraint on γ and λ:

$$\gamma = 2g^2\lambda \,. \tag{81}$$

With the choice in eq.(81) it turns out that $p^2 = 0$ is a zero of Σ of order four, while $p^2 = g^2 m^2$ is a zero of order eight. It can be seen that if we impose the degeneracy of the second and third pole in eq.(80) their common value is uniquely given by $p^2 = g^2 m^2$.

If eq.(81) holds, diagonalization of the quadratic form in eq.(78) can be achieved by a local field redefinition. For $\gamma = 2g^2\lambda$ eq.(78) becomes

$$\begin{aligned} S_{\mathrm{embed,II,A-b}} &= \int d^4x \Bigg[\sum_{a=1}^{3} \frac{1}{2} b^a_\mu \Big(\lambda^2 (\Box + g^2 m^2)^2 g^{\mu\nu} \\ &\qquad + (1+2\lambda)(\Box + g^2 m^2)\partial^\mu \partial^\nu \Big) b^a_\nu \\ &\qquad + \sum_{a=1}^{3} \frac{1}{2} A^a_\mu (\Box + g^2 m^2) A^{\mu a} \\ &\qquad - \lambda g m \sum_{a=1}^{3} A^a_\mu (\Box + g^2 m^2) b^{\mu a} \Bigg] . \end{aligned} \tag{82}$$

The diagonalization is obtained by setting

$$A_{a\mu} = A'_{a\mu} + \lambda \, gm \, b_{a\mu} \, . \tag{83}$$

This yields finally

$$\begin{aligned} S_{\mathrm{embed,II,A-b}} &= \int d^4x \Bigg[\sum_{a=1}^{3} \frac{1}{2} b_{\mu a} \Big(\lambda^2 (\Box + g^2 m^2) \Box g^{\mu\nu} \\ &\qquad + (1+2\lambda)(\Box + g^2 m^2)\partial^\mu \partial^\nu \Big) b_{\nu a} \\ &\qquad + \frac{1}{2} \sum_{a=1}^{3} A'_{\mu a} (\Box + g^2 m^2) A'^{\mu}_{a} \Bigg] . \end{aligned} \tag{84}$$

Without the condition in eq.(81) it turns out that it is not possible to diagonalize eq.(78) by a local field redefinition. Therefore the physical interpretation of the corresponding model is much less transparent and will not be pursued here.

The UV and IR assignments in eqs.(61), (65) and (69) are also valid for the special choice $\gamma = 2g^2\lambda$, $\alpha = -1$.

A comment is in order here. The off-diagonal terms in the second line of eq.(84) disappear if one further chooses

$$\lambda = -\frac{1}{2}\,. \tag{85}$$

However the condition of the absence of poles at negative values in the propagator of b^0_μ yields an exclusion region for λ given by $-1 < \lambda < 0$, thus forbidding the choice in eq.(85).

In the limit $\lambda \to 0$ we recover under the formal identification $\phi = Db$ the classical Higgs model in the absence of the quartic potential, as it can be seen from eq.(51) and eq.(52). In this limit the scalar ∂b^0, corresponding to the physical Higgs field, is massless. On the other hand in the limit $\lambda \to -1$ the scalar ∂b^0 becomes infinitely massive.

Since $S_{\rm embed,II,A-b}$ in eq.(84) contains the dipole field b^a_μ, a consistent analysis of the asymptotic states requires the use of a second order formalism relying on the introduction of suitable auxiliary fields [28].

Two different formulations are possible. In the first one we rewrite eq.(84) as follows (for notational convenience we suppress the sum over the index a):

$$\begin{aligned} S_1 \;\; = \;\; & \int d^4x \, \Big(\frac{1}{2} A'^{\mu}_a (\Box + g^2 m^2) A'_{a\mu} + h_{a\mu} [C^\mu_a - (\Box + g^2 m^2) b^\mu_a] \\ & + \frac{1}{2} C_{a\mu} [\lambda^2 \Box b^\mu_a + (1 + 2\lambda) \partial^\mu (\partial b_a)] \Big) \,. \end{aligned} \tag{86}$$

The equations of motion for $h_{a\mu}, C_{a\mu}$ and $b_{a\mu}$ are

$$\begin{aligned} & \frac{\delta S_1}{\delta h_{a\mu}} = C^\mu_a - (\Box + g^2 m^2) b^\mu_a = 0 \,, \\ & \frac{\delta S_1}{\delta C_{a\mu}} = h^\mu_a + \frac{1}{2} [\lambda^2 \Box b^\mu_a + (1 + 2\lambda) \partial^\mu (\partial b_a)] = 0 \,, \\ & \frac{\delta S_1}{\delta b_{a\mu}} = -(\Box + g^2 m^2) h^\mu_a + \frac{1}{2} \lambda^2 \Box C^\mu_a + \frac{1}{2} (1 + 2\lambda) \partial^\mu (\partial C_a) = 0 \,. \end{aligned} \tag{87}$$

By going on-shell in eq.(86) with $h_{a\mu}$ S_1 reduces to $S_{\rm embed,II,A-b}$ in eq.(84).

The second possibility is to consider

$$\begin{aligned} S_2 \;\; = \;\; & \int d^4x \, \Big(\frac{1}{2} A'^{\mu}_a (\Box + (gm)^2) A'_{a\mu} + \tilde{h}_{a\mu} (\tilde{C}^\mu_a - \lambda^2 \Box b^\mu_a - (1 + 2\lambda) \partial^\mu (\partial b_a)) \\ & + \frac{1}{2} \tilde{C}_{a\mu} (\Box + (gm)^2) b^\mu_a \Big) \,. \end{aligned} \tag{88}$$

The equations of motion are

$$
\begin{aligned}
\frac{\delta S_2}{\delta \tilde{h}_{a\mu}} &= \tilde{C}^\mu_a - \lambda^2 \Box b^\mu_a - (1+2\lambda)\partial^\mu(\partial b_a) = 0\,, \\
\frac{\delta S_2}{\delta \tilde{C}_{a\mu}} &= \tilde{h}^\mu_a + \frac{1}{2}(\Box + (gm)^2) b^\mu_a = 0\,, \\
\frac{\delta S_2}{\delta b_{a\mu}} &= -(\lambda^2 \Box \tilde{h}^\mu_a + (1+2\lambda)\partial^\mu(\partial \tilde{h}_a)) + \frac{1}{2}(\Box + (gm)^2)\tilde{C}^\mu_a = 0\,. \qquad (89)
\end{aligned}
$$

By going on-shell in eq.(88) with $\tilde{h}_{a\mu}$ S_2 reduces to $S_{\mathrm{embed,II,A-b}}$ in eq.(84).

We notice that in view of the first and the second of eqs.(87) and of the first and the second of eqs.(89) the following relations hold

$$
\tilde{C}^\mu_a = -2h^\mu_a\,, \qquad \tilde{h}^\mu_a = -\frac{1}{2}C^\mu_a\,. \qquad (90)
$$

Each of the choices in eq.(86) and (88) uniquely determines the set of ghosts and antighosts of the model in the second order formalism, as explained in Appendix A. This in turn leads to the identification of two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, both invariant under the S-matrix. The construction of $\mathcal{H}_{1,2}$ is also reported in Appendix A.

It is found that $\mathcal{H}_1$ is spanned by the four components of $A^{'}_{a\mu}$ and by the transverse components of $C_{a\mu}$, while $\mathcal{H}_2$ is spanned by the transverse components of $A^{'}_{a\mu}$ and by the transverse components of $\tilde{C}_{a\mu}$. We identify the physical Hilbert space of the theory as the intersection

$$
\mathcal{H} = \mathcal{H}_1 \cap \mathcal{H}_2\,. \qquad (91)
$$

In view of the relations in eq.(90) we conclude that $\mathcal{H}$ is spanned by the transverse components of $A'_{a\mu}$ only.

In the b^0_μ-sector there are no physical states, as proven in Appendix B. Hence $\mathcal{H}$ is eq.(91) represents the physical Hilbert space of the full minimal embedding model. This is consistent with the analysis of the off-shell cohomology given in sect. 3.3.

4 Conclusions

In this paper we have discussed a mechanism to generate a mass for non-Abelian gauge bosons, based on a fixing condition of an additional shift

symmetry of a larger model, into which massive Yang-Mills theory can be embedded.

In the embedding theory all the fields, besides the gauge fields A^a_μ and their gauge ghosts ω^a, pair into BRST doublets together with their shift ghosts partners. Therefore these additional fields do not affect the cohomology classes of the BRST differential s under which the embedding model is invariant. As a consequence, the cohomology classes of s in the FP-neutral sector (physical observables) reduce to local gauge-invariant quantities constructed only from the field strength and its covariant derivatives.

We have shown that the embedding procedure fails to produce a power-counting renormalizable theory when applied to the field content of the ordinary Higgs model with a single Higgs multiplet.

Nevertheless such an analysis naturally suggests the field content of a possible successful embedding of massive Yang-Mills theory, which we have called the minimal embedding model. In this latter theory only one additional vector field b_μ appears in the bosonic sector together with the gauge fields A^a_μ.

The minimal embedding model fulfills both the UV and the IR power-counting conditions and is therefore renormalizable by power-counting. It also shows a simple cohomological structure in the extended field sector. In this theory spontaneous gauge symmetry breaking is not realized, as a consequence of the fixing condition in eq.(52).

We have discussed the spectrum of the classical action and we have analyzed which constraints have to be imposed on the parameters λ, γ, entering in eq.(52), in order to obtain a set of massive non-Abelian gauge bosons.

In view of the cohomological structure of the relevant embedding BRST differential, the physical observables of the minimal embedding theory are given by the gauge-invariant local quantities, constructed only from the field strength $F^a_{\mu\nu}$ and its covariant derivatives. The physical asymptotic states are the transverse polarizations of the massive fields A'^a_μ. The minimal embedding model could henceforth be regarded as a candidate for a renormalizable model of massive Yang-Mills theory.

Many more checks and investigations are needed in order to settle its status as a field-theoretical description of pure massive gauge bosons. We only hint at some of them here. The classical action in eq.(70) depends on a parameter λ, which can take values outside the region $-1 < \lambda < 0$. In the limit $\lambda \to 0$ the Higgs model without quartic potential is formally recovered,

while in the limit $\lambda \to -1$ the scalar field ∂b^0 acquires infinite mass.

The parameter λ cannot be fixed on the basis of symmetry arguments relying on the structure of the BRST differential. The question of the dependence of the physical observables on λ is a subject deserving further study, in order to explore the physical predictions of the model and to compare them with those of other theories allowing for massive gauge bosons. Closely related to this issue, the inclusion of massive fermions is also worthwhile to be analyzed, in order to shed light on possible phenomenological applications of the minimal embedding formalism.

It may also happen that the model discussed in this paper could suggest other yet unexplored possibilities to provide a field-theoretical framework for the description of massive Yang-Mills theory.

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A Physical states in the $A'_{a\mu} - b_{c\nu}$-sector

In this Appendix we construct the physical Hilbert spaces $\mathcal{H}_{1,2}$ identified respectively by the choice in eq.(86) (type-I second order formalism) and in eq.(88) (type-II second order formalism).

Each choice uniquely defines the set of ghost fields of the model in the second order formalism. Once the set of ghost fields is constructed, the physical states can be selected as the non-trivial cohomology classes of the relevant linearized BRST differential $\hat{s}$.

The analysis carried out here applies at the classical level. At higher orders the relevant BRST charge, acting on the asymptotic states, can differ from the linearized BRST differential $\hat{s}$ by factors related to wave-function renormalizations and possible mixings among the fields [9]. The kernel of the proof remains however unaffected and can be extended without additional difficulties to the full renormalized minimal embedding model.

A.1 Type-I second order formalism

In order to define the ghost content of the model in the second-order formalism the first step is to reconstruct the squares from eq.(86). For this purpose we use eq.(83) to express $A'_{a\mu}$ as a function of $A_{a\mu}$ and $b_{a\mu}$:

$$A'_{a\mu} = A_{a\mu} - \lambda g m b_{a\mu} \,. \tag{92}$$

By substituting eq.(92) into eq.(86) and by using the relations

$$\begin{aligned} C^{\mu}_{a} &= (\Box + g^2 m^2) b^{\mu}_{a} \,, \\ \partial C_a &= (\Box + (gm)^2) \partial b \,, \end{aligned} \tag{93}$$

(a consequence of the first of eqs.(87)) we get [4]

$$\begin{aligned} S_1 \;\; = \;\; & \int d^4x \, \Bigg(\frac{1}{2} A_{a\mu} \Box A^{\mu}_{a} + \frac{1}{2} (\partial A_a)^2 \\ & + \frac{1}{2} (\lambda C_{a\mu} - gm A_{a\mu} + \partial_\mu (\partial b_a))^2 \\ & + h_{a\mu} [C^{\mu}_{a} - (\Box + g^2 m^2) b^{\mu}_{a}] \\ & - \frac{1}{2} (\partial A_a + gm \partial b_a)^2 \Bigg) \,. \end{aligned} \tag{94}$$

The terms in the first line are the bilinear contributions from $-\frac{1}{4}(F^{a}_{\mu\nu})^2$. The structure of the ghost system is completely determined by eq.(94). The relevant linearized BRST transformations are

$$\begin{aligned} & \hat{s} A'_{\mu a} = \partial_\mu \omega_a - \lambda g m \xi_{a\mu} \,, \qquad \hat{s} \omega_a = 0 \,, \\ & \hat{s} \bar{\omega}_a = \partial A_a + g m \partial b_a \,, \\ & \hat{s} \bar{\theta}_{a\mu} = h_{a\mu} \,, \qquad \hat{s} h_{a\mu} = 0 \,, \qquad \hat{s} C_{a\mu} = \theta_{a\mu} \,, \qquad \hat{s} \theta_{a\mu} = 0 \,, \\ & \hat{s} \psi_{a\mu} = \lambda C_{a\mu} - g m A_{a\mu} + \partial_\mu \partial b_a \,, \\ & \hat{s} b_{a\mu} = \xi_{a\mu} \,, \qquad \hat{s} \xi_{a\mu} = 0 \,. \end{aligned} \tag{95}$$

The complete action S_1, including the ghost-dependent terms, is

$$S_1 \;\; = \;\; \int d^4x \, \Bigg(\frac{1}{2} A_{a\mu} \Box A^{\mu}_{a} + \frac{1}{2} (\partial A_a)^2$$

[4] For notational convenience we do not write explicitly the sum over the index a.

$$
\begin{aligned}
&+\frac{1}{2}(\lambda C_{a\mu} - gmA_{a\mu} + \partial_\mu(\partial b_a))^2 + h_{a\mu}[C_a^\mu - (\Box + g^2m^2)b_a^\mu] \\
&-\frac{1}{2}(\partial A_a + gm\partial b_a)^2 \\
&-\psi_{a\mu}[\lambda\theta_a^\mu - gm\partial^\mu\omega_a + \partial^\mu(\partial\xi_a)] - \bar{\theta}_{a\mu}[\theta_a^\mu - (\Box + g^2m^2)\xi_a^\mu] \\
&+\bar{\omega}_a(\Box\omega_a + gm\partial\xi_a)) \,. \qquad (96)
\end{aligned}
$$

The fields of the model are $A'_{a\mu}, h_{a\mu}, C_{a\mu}$. Therefore we need to eliminate $b_{a\mu}$ in favour of $h_{a\mu}$ and $C_{a\mu}$. By using the first and the second of eqs.(87) we obtain

$$
b_a^\mu = \frac{1}{g^2m^2}\left[C_a^\mu - \frac{1}{\lambda^2}\left(-2h_a^\mu - \frac{(1+2\lambda)}{g^2m^2}\partial^\mu\left(\partial C_a + \frac{2}{(1+\lambda)^2}\partial h_a\right)\right)\right] \,, \quad (97)
$$

valid for $\lambda \neq -1$. By taking the $\hat{s}$-variation of both sides of eqs.(97) we get a constraint on ξ_a^μ:

$$
\xi_a^\mu = \frac{1}{g^2m^2}\left[\theta_a^\mu + \frac{1+2\lambda}{\lambda^2}\frac{1}{g^2m^2}\partial^\mu(\partial\theta_a)\right] \,. \qquad (98)
$$

This is valid for $\lambda \neq -1$. For $\lambda \neq -1$ it can be proven that it is equivalent to the constraint

$$
\hat{s}h_a^\mu = \lambda^2\Box\xi_a^\mu + (1+2\lambda)\partial^\mu(\partial\xi_a) = 0 \qquad (99)
$$

The latter expression is however valid also for $\lambda = -1$.

We can now move to the analysis of

$$
\mathcal{H}_1 = \text{Ker } \hat{s}/\text{Im } \hat{s} \,. \qquad (100)
$$

By imposing the equations of motion in the ghost sector with the constraint in eq.(98) one finds

$$
\xi_a^\mu = \frac{1}{\lambda gm}\partial^\mu\omega_a \,, \qquad \theta_a^\mu = \frac{1}{\lambda}gm\partial^\mu\omega_a \,, \qquad (101)
$$

with

$$
\Box\omega_a = 0 \,. \qquad (102)
$$

From eq.(95) the BRST transformations in the bosonic sector become

$$
\hat{s}A'_{a\mu} = 0 \,, \qquad \hat{s}C_{a\mu} = \frac{1}{\lambda}gm\partial_\mu\omega_a \,, \qquad \hat{s}h_{a\mu} = 0 \,. \qquad (103)
$$

Moreover $h_{a\mu}$ is $\hat{s}$-exact (since $\hat{s}\psi_{a\mu} = h_{a\mu}$) and $\bar{\omega}_a$ forms a doublet with $\partial A_a + gm\partial b_a$. ξ_a^μ forms a doublet with b_a^μ given in eq.(97). We conclude from the above remarks and from eq.(103) that the space $\mathcal{H}_1$ in eq.(100) is spanned by the four components of $A'_{a\mu}$ and by the transverse components of $C_{a\mu}$.

A.2 Type-II second order formalism

As before we start by reconstructing the squares from eq.(88). We use eq.(92) and the first of eqs.(89) to rewrite S_2 in eq.(88) as follows:

$$
\begin{aligned}
S_2 = \int d^4x \Big(& + \frac{1}{2} A_{a\mu} \Box A_a^\mu + \frac{1}{2} (\partial A_a)^2 \\
& + \tilde{h}_{a\mu} (\tilde{C}_a^\mu - \lambda^2 \Box b_a^\mu - (1+2\lambda) \partial^\mu (\partial b_a)) \\
& + \frac{1}{2} \Big(\frac{1}{\lambda} \tilde{C}_a^\mu + \lambda (gm)^2 b_a^\mu - gm A_a^\mu - \frac{1+\lambda}{\lambda} \partial^\mu (\partial b_a) \Big)^2 \\
& - \frac{1}{2} (\partial A_a + gm \partial b_a)^2 \Big) \, .
\end{aligned} \tag{104}
$$

Eq.(104) fixes the linearized BRST transformations of the fields:

$$
\begin{aligned}
& \hat{s} A'_{a\mu} = \partial_\mu \tilde{\omega}_a - \lambda gm \tilde{\xi}_{a\mu} \, , \qquad \hat{s} \tilde{\omega}_a = 0 \, , \\
& \hat{s} \tilde{\bar{\omega}}_a = \partial A_a + gm \partial b_a \, , \\
& \hat{s} \tilde{\psi}_{a\mu} = \frac{1}{\lambda} \tilde{C}_{a\mu} + \lambda (gm)^2 b_{a\mu} - gm A_{a\mu} - \frac{1+\lambda}{\lambda} \partial_\mu (\partial b_a) \, , \\
& \hat{s} \tilde{\bar{\theta}}_{a\mu} = \tilde{h}_{a\mu} \, , \qquad \hat{s} \tilde{h}_{a\mu} = 0 \, , \\
& \hat{s} \tilde{C}_{a\mu} = \tilde{\theta}_{a\mu} \, , \qquad \hat{s} \tilde{\theta}_{a\mu} = 0 \, , \\
& \hat{s} b_{a\mu} = \tilde{\xi}_{a\mu} \, , \qquad \hat{s} \tilde{\xi}_{a\mu} = 0 \, .
\end{aligned} \tag{105}
$$

The complete action S_2, including the ghost-dependent terms, is

$$
\begin{aligned}
S_2 = \int d^4x \Big(& + \frac{1}{2} A_{a\mu} \Box A_a^\mu + \frac{1}{2} (\partial A_a)^2 \\
& + \tilde{h}_{a\mu} (\tilde{C}_a^\mu - \lambda^2 \Box b_a^\mu - (1+2\lambda) \partial^\mu (\partial b_a)) \\
& + \frac{1}{2} \Big(\frac{1}{\lambda} \tilde{C}_a^\mu + \lambda (gm)^2 b_a^\mu - gm A_a^\mu - \frac{1+\lambda}{\lambda} \partial^\mu (\partial b_a) \Big)^2 \\
& - \frac{1}{2} (\partial A_a + gm \partial b_a)^2
\end{aligned}
$$

$$
\begin{aligned}
&- \tilde{\psi}_{a\mu}(\frac{1}{\lambda}\tilde{\theta}^\mu_a + \lambda(gm)^2\tilde{\xi}^\mu_a - gm\partial^\mu\tilde{\omega}_a - \frac{1+\lambda}{\lambda}\partial^\mu(\partial\tilde{\xi}_a)) \\
&+ \tilde{\bar{\omega}}_a(\Box\tilde{\omega}_a + gm\partial\tilde{\xi}_a) \\
&- \tilde{\bar{\theta}}_{a\mu}(\tilde{\theta}^\mu_a - \lambda^2\Box\tilde{\xi}^\mu_a - (1+2\lambda)\partial^\mu(\partial\tilde{\xi}_a))\Big) \, .
\end{aligned} \tag{106}
$$

Again we eliminate $b_{a\mu}$ in favour of $\tilde{C}_{a\mu}$ and $\tilde{h}_{a\mu}$. By using the first and the second of eqs.(89) we get

$$
b^\mu_a = -\frac{2}{(gm)^2}\Big[\tilde{h}^\mu_a + \frac{1}{2\lambda^2}\Big(\tilde{C}^\mu_a + \frac{(1+2\lambda)}{(gm)^2}\partial^\mu\Big(\frac{1}{(1+\lambda)^2}\partial\tilde{C}_a + 2\partial\tilde{h}_a\Big)\Big)\Big] \, , \tag{107}
$$

valid for $\lambda \neq -1$. By taking the $\hat{s}$-variation of both sides of eqs.(107) we get a constraint on $\tilde{\xi}^\mu_a$:

$$
\tilde{\xi}^\mu_a = -\frac{2}{(gm)^2}\Big[\frac{1}{2\lambda^2}\Big(\tilde{\theta}^\mu_a + \frac{1+2\lambda}{(1+\lambda)^2}\frac{1}{(gm)^2}\partial^\mu(\partial\tilde{\theta}_a)\Big)\Big] \, . \tag{108}
$$

This condition is valid for $\lambda \neq -1$. For $\lambda \neq -1$ it can be shown to be equivalent to the constraint

$$
\hat{s}\tilde{h}^\mu_a = -\frac{1}{2}(\Box + (gm)^2)\tilde{\xi}^\mu_a = 0 \, . \tag{109}
$$

The latter expression is however valid also for $\lambda = -1$.

The equations of motion in the ghost sector admit a solution for $\tilde{\xi}_{a\mu}$ in the same class as in eq.(101), given by

$$
\tilde{\xi}_{a\mu} = -\frac{1}{gm}\partial_\mu\tilde{\omega}_a \, . \tag{110}
$$

The constraint in eq.(109) implies

$$
(\Box + (gm)^2)\tilde{\omega}_a = 0 \, . \tag{111}
$$

$\tilde{\theta}_{a\mu}$ is then given by

$$
\tilde{\theta}_{a\mu} = (1+\lambda)^2(gm)\partial_\mu\tilde{\omega}_a \, . \tag{112}
$$

The linearized BRST transformations in the bosonic sector hence become

$$
\begin{aligned}
&\hat{s}A'_{a\mu} = (1+\lambda)\partial_\mu\tilde{\omega}_a \, , \\
&\hat{s}\tilde{C}_{a\mu} = (1+\lambda)^2 gm\partial_\mu\tilde{\omega}_a \, , \\
&s\tilde{h}_{a\mu} = 0 \, .
\end{aligned} \tag{113}
$$

Moreover $\tilde{h}_{a\mu}$ is $\hat{s}$-exact (since $\hat{s}\tilde{\psi}_{a\mu} = \tilde{h}_{a\mu}$) and $\tilde{\bar{\omega}}_a$ forms a doublet with $\partial A_a + gm\partial b_a$. $\tilde{\xi}^a_\mu$ forms a doublet with b^a_μ in eq.(107). The space

$$\mathcal{H}_2 = \text{Ker } \hat{s}/\text{Im } \hat{s} \tag{114}$$

is spanned by the transverse components of $A'_{a\mu}$ and of $\tilde{C}_{a\mu}$,

B Physical states in the b^0_μ-sector

In the second order formalism the relevant quadratic part in the b^0_μ-sector for the action given in eq.(70), including the ghost-dependent terms, is

$$\begin{aligned} S_0 \ &= \int d^4x \Big(-\frac{1}{2}C^0_\mu C^{0\mu} - C^0_\mu(\lambda(\Box + (gm)^2)b^{0\mu} + \partial^\mu(\partial b^0)) \\ &\qquad + \psi^0_\mu(\lambda(\Box + (gm)^2)\xi^{0\mu} + \partial^\mu(\partial\xi^0))\Big) . \end{aligned} \tag{115}$$

The equations of motion in the bosonic sector are

$$\begin{aligned} \frac{\delta S_0}{\delta C^0_\mu} \ &= \ -C^0_\mu - \lambda(\Box + (gm)^2)b^0_\mu - \partial_\mu(\partial b^0) = 0 , \\ \frac{\delta S_0}{\delta b^0_\mu} \ &= \ -\lambda(\Box + (gm)^2)C^0_\mu - \partial_\mu(\partial C^0) = 0 . \end{aligned} \tag{116}$$

From the first of eqs.(116) we get

$$C^0_\mu = -\lambda(\Box + (gm)^2)b^0_\mu - \partial_\mu(\partial b^0) . \tag{117}$$

From the second of eqs.(116) we get

$$\lambda(\Box + (gm)^2)C^0_\mu + \partial_\mu(\partial C^0) = 0 . \tag{118}$$

By taking the divergence of eq.(118) we find

$$(1 + \lambda)\Box\partial C^0 + \lambda(gm)^2\partial C^0 = 0 . \tag{119}$$

The scalar ∂C^0 has mass $p^2 = \frac{\lambda}{1+\lambda}(gm)^2$. It goes to infinity for $\lambda \to -1$.

The relevant linearized BRST transformations are

$$\hat{s}\psi^0_\mu = C^0_\mu , \quad \hat{s}C^0_\mu = 0 , \quad \hat{s}b^0_\mu = \xi^0_\mu , \quad \hat{s}\xi^0_\mu = 0 . \tag{120}$$

In view of eq.(120) we see that there are no additional physical particles in the b^0_μ-sector.

We wish to make a comment on the limit $\lambda \to -1$. In this limit we see from the equation of motion of ψ^0_μ that

$$\partial\xi_0 = 0\,. \tag{121}$$

Therefore the scalar ∂b_0 would be physical (since $\hat{s}(\partial b_0) = \partial\xi_0 = 0$) However, in the limit $\lambda \to -1$ we get from eq.(117) that

$$\partial b^0 = \frac{1}{(gm)^2}\partial C^0\,. \tag{122}$$

The scalar ∂C^0 in turn becomes infinitely massive in the limit $\lambda \to -1$ and therefore it disappears from the physical spectrum.

C UV and IR degrees of the minimal embedding model

Field	UV dim. (δ)	IR dim. (ρ)
$A_{a\mu}$	1	1
$b_{0\mu}$	0	2
$b_{a\mu}$	0	1
$\psi_{0\mu}$	1	2
$\psi_{a\mu}$	1	2
$\xi_{0\mu}$	1	2
$\xi_{a\mu}$	1	2
$\bar{\omega}^a$	1	1
ω^a	1	1

Table 1 - UV and IR assignments for the fields of the minimal embedding model.

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